XIII. *Results of the Comparisons of the Standards of Length of England, Austria, Spain, United States, Cape of Good Hope, and of a second Russian Standard, made at the Ordnance Survey Office, Southampton. By Lieutenant-Colonel* A. R. Clarke, *C.B., R.E., F.R.S., &c., under the direction of Major-General Sir* Henry James, *R.E., F.R.S., &c., Director-General of the Ordnance Survey. With a Preface and Notes on the Greek and Egyptian Measures of Length by Sir* Henry James.

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The following account of the results of the Comparisons of the Standards of Length of England, Austria, Spain, United States, Cape of Good Hope, and of a second Russian Standard at the Ordnance Survey Office has been drawn up by Lieutenant-Colonel Clarke, and is a sequel to the abstract of the results of the Comparisons of the Standards of Length of England, France, Belgium, Prussia, Russia, India, and Australia which the Royal Society has done us the honour to publish in the Philosophical Transactions for 1867, vol. clvii. p. 161.

The accurate determination of the lengths of the various standards employed by so many nations in the measure of the bases of their triangulations, which are now being united into one vast network of triangles, covering the whole of Europe, can scarcely fail to be of great importance for the advancement of physical science. To the comparison of these lengths I have added the result of our endeavours to recover the correct lengths of the most ancient measures of length with which we are acquainted, viz. those of Ancient Egypt, not only because our own measures are obviously derived from them, but also because we thus obtain the accurate relative value of the measures and distances given in the most ancient works on Astronomy and Geodesy which have come down to us.

The Ancient Egyptians employed two measures of length, viz. the common and the royal cubits.

1st. As regards the *common cubit*, we have the statement of Herodotus that the Egyptian cubit was equal to the Greek cubit, "that of Samos;" and we learn from the measurements of the Hecatompedon at Athens, by Mr. Penrose, that the Greek foot was equal to 1·013 foot, or 12·156 inches, and consequently the Greek cubit was equal to 1·520 foot, or 18·240 inches.

2nd. The most recent measures of the base of the First or Great Pyramid, that of King Cheops, viz. those made by the Royal Engineers and Mr. Inglis, a civil engineer, give a mean length of 9120 inches, or 500 cubits of 18·240 inches for the side of the square base, or 750 Egyptian feet, each Egyptian foot being equal to 1·013 English foot.

3rd. The Second Pyramid, according to the measures of Colonel HOWARD VYSE and Mr. PERRING, has a base of 707·5 feet square, or 700×1·011 feet.

4th. The Third Pyramid has a base, according to VYSE and PERRING, of 354·5 feet, or 350 Egyptian feet square, of 1·013 English foot exactly.

We may therefore confidently assume that 1·013 foot was the true length both of the ancient Greek and the ancient common Egyptian foot, and that the length of the *common Egyptian cubit* was 18·240 inches.

We have in the British Museum a double *royal cubit*, found in the ruins of the Temple of Karnak in Egypt; and I found its length to be 41·40 inches, and that of the single cubit consequently 20·70 inches, or 1·725 foot.

The pyramid which stands in the middle of the three, before the Great Pyramid (that of the daughter of King CHEOPS), has a base, according to VYSE and PERRING, of 172·5 feet square, and therefore 100 royal cubits square exactly.

But the same authors give the breadths of no less than seven of the passages in the pyramids, including the entrances to the First, Second, and Third Pyramids, all of 41·5 inches (two cubits of 20·750 inches).

DOURSTHER, from the measures of the nilometer at Elephantine and of three or four cubits found in the ruins of Memphis, which almost exactly correspond with each other, estimated the length of the royal cubit at 20·721 inches (see CONDÉE, 'Dictionnaire des Poids et Mesures').

Looking to these facts, and feeling it almost certain that the common and the royal cubit had some definite relation to one another, like that between the link and foot of our own country (66 feet equal 100 links), I infer that the most probable length of the *royal cubit* was 20·727 inches, and that 88 *royal cubits* were equal to 100 *common cubits* of 18·240 inches.

This does not admit of rigid demonstration. The dimensions of VYSE and PERRING seem to be given to the nearest half inch, and the measures of length sold in this country differ from one another as much as the length of the double cubit in the British Museum differs from its estimated length.

HENRY JAMES, *Major-General.*

[NOTE.—Since writing the above I have had an opportunity of consulting Don V. VAZQUEZ QUEIPO's 'Essai sur les Systèmes Métriques &c.,' in which he gives a description of ten royal cubits which have been found from time to time in the ancient buildings of Egypt. He has numbered these from 1 to 10, and given the lengths of four of them, and the mean length of all the Standards found.

		m.		inches.
No. 5.	Stone	=0·52650	or	20·728
8.	Wood	=0·52514	„	20·675
9.	Wood	=0·52598	„	20·708
10.	Wood	=0·52448	„	20·649
Mean length of all the Standards		=0·52500	„	20·669

No. 10 is the royal cubit from the Temple at Karnak, which is in the British Museum, and which I measured, and found to be of the same length as is here given of No. 9. No. 10 is made of wood, as are also Nos. 8 & 9. But No. 5, now in Florence, is made of slate or schist, and a much higher value must be attached to its length than to the lengths of the three wooden ones, because, being of stone, it was not liable to alteration, as those made of wood were, during the 3500 years, at least, which must have elapsed since they were made.

We cannot suppose for a moment that stone cubits were used by the artificers employed in building the ancient temples; the workmen were supplied with the wooden double cubits, of which Nos. 9 & 10 are examples.

These wooden cubits are divided, although apparently with a fine saw, with as much, or even greater accuracy than the generality of the measures with which the workmen of the present day are supplied.

The stone cubits were probably preserved as standards, with the name of the reigning King engraved upon them, and referred to to verify the lengths on the wooden cubits.

The length of this stone, royal cubit, $0^{m}\cdot5265$ or 20·728 inches, corresponds with my previously estimated length, 0·52646 = 20·727; and the length of the common cubit is 0·46329 = 18·240.

The royal cubit is divided into 7 palms of 2·961 inches, the common cubit into 6 palms of 3·040 inches; and the mean is exactly 3 inches: the length of the English foot is consequently equal to the mean length of the ancient Egyptian common and royal foot, although it probably is so by accident only.—H. J., August 1873.]

Comparison Apparatus.

The Comparisons of Standards to be recorded in the present paper have been made in the same room, and with generally the same apparatus, as those described in a previous paper read before the Royal Society, December 13, 1866. In that paper were given the results of the Comparisons of the Standards of England, France, Belgium, Prussia, Russia, India, and Australia; in the present paper the results will be given for the Standards of Austria, Spain, America, the Cape of Good Hope, and a second Russian Standard.

The general method of conducting the comparisons explained in the former paper has been adhered to with no material alterations. No kind of unsteadiness in the piers carrying the microscopes has at any time been noticed, nor in the mahogany beam carrying the bars under observation. The insulation of the flooring has remained perfect.

Some little improvements, however, have been made. The candles were originally held in wooden conical holders, which rested immediately upon the cast-iron plates holding the microscopes. In moving the candle-holder, so as to keep the centre of the image of the flame on the dot, it seemed possible that a disturbance might be communicated to the cast-iron microscope-holders. In order to remove this possibility a new form of candle-holder was introduced, which does not come into contact with the cast-iron plates, and admits of being very easily adjusted to the proper position from time to time. Some improvement has been also made in the apparatus for measuring calibration errors of thermometers, and also (mainly by increasing the quantity of water) in that for comparisons of thermometers.

The values of one micrometer-division of the microscopes H and K obtained in 1863 were

$$h=0\cdot79494\pm0\cdot00008,$$

$$k=0\cdot79800\pm0\cdot00009;$$

in 1865,

$$h=0\cdot79566\pm0\cdot00008,$$

$$k=0\cdot79867\pm0\cdot00009.$$

In 1869 the values were determined again with the result,

$$h=0\cdot79564\pm0\cdot00010,$$

$$k=0\cdot79837\pm0\cdot00010.$$

Two new micrometer-microscopes marked I and J have been added to the apparatus. They are longer than H and K; the distance from the object observed to the object-glass is 4 inches, from the object-glass to the micrometer-threads 14·6 inches. The value of one division of either of these microscopes is very little different from that of one division in the microscopes H and K. They were determined from ten measurements of the same space, giving it equal to $1368\cdot45i$ or $1381\cdot50j$, where i and j represent the values of one division in the respective microscopes. From these measures we get

$$i=0\cdot81032\pm0\cdot00018,$$

$$j=0\cdot80270\pm0\cdot00015.$$

These microscopes have never been used for any large measurements, and the values are therefore sufficiently precise.

The space between the lines on the contact-apparatus has been repeatedly examined, and appears to be as nearly as possible invariable. The results obtained from very numerous comparisons in different years are:—

In 1863	$\delta=565\cdot62$,
„ 1864	$\delta=565\cdot55$,
„ 1865	$\delta=565\cdot69$,
„ 1869	$\delta=565\cdot64$.

The Klafter.

Two copies of the Vienna Klafter have been compared at Southampton, viz. the Pulkowa copy and the Milan copy. In order to effect the measurement of the klafter, it was most convenient to construct, in the first instance, a bar of the length of half a klafter, to determine its length by comparison with the standard yard and standard foot, and then to compare the klafters with two lengths of this bar.

The half klafter was constructed by Messrs. TROUGHTON and SIMS; it is of Swedish iron 38·3 inches in length, 1·0 in breadth, and 0·475 inch in depth. On its upper surface it bears four raised disks, marked *a*, *b*, *c*, *d*; the distance $ab=1·33$ inch, $bc=4·00$ inches, and $bd=36$ inches; thus the distance between the transverse lines on *a* and *d*, marking the total length of the half klafter, is 37·33 inches approximately. The necessity for the line *c* arises from the circumstance that the small space *a b* could not be compared with a corresponding space on the standard foot, as two micrometer-microscopes cannot be fitted up at so small a distance apart; consequently *a b* is determined as the difference between the spaces $ac=5·33$ inches and $bc=4·00$, which two spaces were compared with corresponding spaces on the standard foot. We shall designate this bar by the symbol $K_{\frac{1}{2}}$.

For convenience, and to save reference to the former paper, it may be well to explain that the standard foot has on its upper surface thirteen division-lines marked *a*, *b*, *c*, *d*, *e*, *f*, *g*, *h*, *k*, *l*, *m*, *n*, *p*, marking the twelve inches. The first inch (*a b*) is subdivided into tenths by lines marked (from *a* towards *b*) 1, 2, 3, 4, 5, 6, 7, 8, 9. The tenths [2 . 3] and [6 . 7] are subdivided again into tenths. In the former of these spaces falls the toise-line τ, and in the second the metre-line μ. The line by which the length of the half klafter is determined also falls in [6 . 7]; it is the seventh line from 6 towards 7, or the third line from 7 towards 6. The error of this line with reference to the lines 6 and 7 is, the unit being the one millionth of a yard,

$$x_k = -4·29 \pm 0·085;$$

that is to say, if [6 . *k*] be the distance from the line 6 to the line *k*,

$$[6\,.\,k] = \tfrac{7}{10}[6\,.\,7] + x_k.$$

The space [*a c*] of the half klafter was compared with the space [*k g*] on the standard foot. We therefore require to have the probable error of [*k g*]. Now [*k g*]=[*a g*]−[*a k*]; and expressed in terms of the errors of the other lines the value of [*k g*] is

$$[kg] = \frac{533}{100}\mathrm{I} - \frac{3x_6}{10} - x_k - \frac{7x_7}{10} - \frac{67x_b}{100} + \frac{533x_g}{600},$$

where $x_6\, x_7$ are the errors of the lines 6 . 7 with respect to the lines *a b*, and $x_b\, x_g$ the errors of the lines *b g* with respect to the lines *a p*. Also $\mathrm{I}=\frac{1}{12}$ of the standard foot F, From this expression we can at once determine the probable error of [*k g*]; we shall not enter into the details of the calculation, but simply give the result, namely,

$$[kg] = \tfrac{533}{100}\mathrm{I} + 1·40 \pm 0·108.$$

This is on the supposition of the klafter and the yard being both at the temperature of 62°.

The standard temperature of the Vienna klafter is 61°·25 Fahrenheit, or 16°·25 Centigrade, and this temperature is therefore adopted for the half klafter; consequently we require the length of [*k g*] at the same temperature. Now the comparisons made between the standard foot and the standard yard Y_{55} give this result,

$$F=\tfrac{1}{3}Y_{55}-0\cdot36\pm0\cdot104,$$

the foot and yard being both at 61°·25. Hence, at the same temperature,

$$[k\,g]=\tfrac{533}{3600}Y_{55}+1\cdot24\pm0\cdot118.$$

The space [*b c*] of 4 inches on the half klafter was compared with the space [*c g*] of 4 inches on the standard foot. The value of this space is, both the foot and the yard being at 61°·25,

$$[c\,g]=\tfrac{4}{36}Y_{55}-2\cdot73\pm0\cdot062.$$

The space [*b d*] of 36 inches was compared with the standard yard Y_{55} twenty times between March 19th and 27th, 1866, at temperatures varying from 41°·6 to 43°·7—sixteen times between June 25th and July 2nd, the temperature varying from 62°·7 to 67°·4—and six times in September 1867, at about 64°,—in all forty-two comparisons.

Reduced by the method of least squares the result is

$$[b\,d]=Y_{55}+0\cdot17+(t-61^\circ\cdot25)\ \cdot0224.$$

The reciprocals of the weights of the determinations of the quantities 0·17 and ·0224 are 0·03204 and 0·00018 respectively; and the sum of the squares of the forty-two errors of observation is

$$\Sigma(e^2)=19\cdot949.$$

Hence we find the probable error of a single comparison to be

$$\pm0\cdot674\sqrt{\frac{19\cdot949}{42-2}}=\pm0\cdot476.$$

The probable error therefore of 0·17 is

$$\pm0\cdot476\sqrt{\cdot03304}=\pm0\cdot085;$$

and at the temperature of 61°·25,

$$[b\,d]=Y_{55}+0\cdot17\pm0\cdot085.$$

The 4-inch space [*b c*] on the half klafter was compared with the space [*c g*] on the foot fifteen times between April 13th and 19th, 1866, at temperatures varying between 47° and 49°—ten times between June 9th and 13th, the temperatures varying from 60° to 62°—and six times in September 1867, at the temperature of 58°.

The 5·33-inch space [*a c*] was compared with the space [*k g*] on the foot ten times in April 1866, at a temperature varying little from 46°—fifteen times in June of the same year, at temperatures varying little from 60°—and five times in September 1867, the temperature being 62° or 63°.

In order to reduce these comparisons, put

$x_{\prime}=[b\,c]-[c\,g]$ . . . . . at 61°·25 Fahr.
$x'=[a\,c]-[k\,g]$ . . . . . „
$y_{\prime}$=expansion of 1 inch of half klafter for 1° Fahr.
y'= „ „ foot „ „

Also for convenience put $4(y_{\prime}-y')=3y$; then the equations of condition take the form

$$x_{\prime}+3a_1y-b_1=0,$$
$$x_{\prime}+3a_2y-b_2=0,$$
.

$$x'+4a'_1y-b'_1=0,$$
$$x'+4a'_2y-b'_2=0,$$
.
.

which give, finally, a system of equations of the form

$$0=31\,x'+3(a)y-(b),$$
$$0=30\,x'+4(a')y-(b'),$$
$$0=3(a)x_{\prime}+4(a')x'+\{9(a^2)+16(a'^2)\}y-3(ab)-4(a'b').$$

The substitution of the numerical values gives

$x_{\prime}=+4·10$ . . . with reciprocal of weight=0·04457,
$x'=-1·86$ . . . „ „ 0·04721,
$y=+0·00043$ „ „ 0·00003.

The sum of the squares of the residual errors of the sixty-one equations is 11·295; hence the probable error of a single comparison is

$$\pm 0·674\sqrt{\frac{11·295}{61-3}}=\pm 0·297;$$

and the lengths of the two small spaces are therefore, both bars being at the temperature of 61°·25, these:—

$$[b\,c]=[c\,g]+4·10\pm 0·063,$$
$$[a\,c]=[k\,g]-1·86\pm 0·065.$$

We now know the values of the different spaces on the half klafter, namely, by the difference of the last two equations,

$$[a\,b]=[k\,c]-5·96,$$

and

$$[b\,d]=Y_{55}+0·17;$$

adding these together,

$$[a\,d]=Y_{55}+[k\,c]-5·79.$$

The difference of the expressions we have given for $[k\,g]$ and $[c\,g]$ gives

$$[k\,c]=\tfrac{133}{3600}\,Y_{55}+3\cdot97\,;$$

consequently the length of the half klafter is

$$[a\,d]=\tfrac{3733}{3600}\,Y_{55}-1\cdot82.$$

In order to determine the probable error of this result, it is necessary to form the algebraic expression for $[a\,d]$, which involves all the observations and operations. It is found to be this:—

$$[a\,d]=\tfrac{3733}{3600}\,Y_{55}+\tfrac{133}{600}\,x_g-\tfrac{67}{100}\,x_b+x_c-\tfrac{3}{10}\,x_6-\tfrac{7}{10}\,x_7-x_k+\tfrac{133}{1200}\,v+x+x'-x_{\prime}.$$

This compound error is formed from seven independent operations. The first term in x_g depends on the subdivision of the standard foot into two 6-inch spaces; the second, involving the terms in x_b and x_c, on the determination of the errors of the different inch lines on one half of the foot; the third, the terms in $x_6\,x_7$, on the division of one of the inches into tenths; the fourth, x_k, on the division of one of the tenths into hundredths; the fifth, in v, on the comparison of the standard foot with the standard yard; the sixth, in x, on the comparison of the standard yard with the half klafter; and the seventh, $x'-x_{\prime}$, on the comparison of the small spaces on the half klafter with the foot. Taking the different component probable errors in this same order, we get the quantity

$$\pm\sqrt{\{(\cdot008)^2+(\cdot045)^2+(\cdot052)^2+(\cdot085)^2+(\cdot011)^2+(\cdot085)^2+(\cdot076)^2\}},$$

which is equal to $\pm0\cdot158$.

The length of Y_{55} at $61^\circ\cdot25$ is (see 'Comparisons of Standards,' page 276) $4\cdot886$ less than its length at 62°, and at 62° it is less than a true yard by $0\cdot40$; hence the length of the half klafter at $61^\circ\cdot25$,

$$K_{\frac{1}{2}}=1\cdot03693714\,\mathfrak{Y}\pm\cdot00000016,$$

where $\mathfrak{Y}$ is a true yard.

During the comparisons the bar was supported on two knife-edges, at one fourth and three fourths of its length. The positions of these supports are marked on the bar.

The Pulkowa Copy of the Klafter.

This is a bar of steel (apparently) 0·55 inch in breadth and 1·06 inch in depth, but not very evenly planed. On its under surface are fixed small brass pieces at one fourth and three fourths of its length, on which it is supported during comparisons. Its upper surface has seven inlaid plates of silver, on each of which is engraved a dot. The extreme dots are one toise apart. Near one of the end dots is the klafter dot. Two other dots mark the half klafter and the half toise. The two remaining dots subdivide the half klafter into *thirds* or Vienna feet. We shall use the symbol K′ for the length of the klafter on this bar.

The klafter and the half klafter were mounted side by side in a box for comparison, their middle points being opposite one another. The microscopes H and K being mounted at the distance of half a klafter apart, the order of observation at each visit was this:—

1st, the thermometers;
2nd, the half klafter;
3rd, the first half of the Vienna klafter;
4th, the second half of the Vienna klafter;
5th, the half klafter;
6th, the thermometers.

The "first half" of the Vienna klafter is that which is on the same side with the toise dot: we shall denote it by $K'_{1.2}$, and the second half by $K'_{2.3}$; so that $K'_{1.2}+K'_{2.3}=K'$.

The klafter and half klafter were compared twenty times in September 1867, at temperatures varying between 61°·3 and 64°·7, and seven times in January 1868, at temperatures varying between 37° and 38°·6.

The reduction of these comparisons by least squares, putting $x+fy$ and $x'+fy$ for the excess of the first and second halves of the klafter over the half klafter at the temperatures $61°·25+f$, leads to a system of equations,

$$0=n(x+x')+2(f)y-(j)-(j'),$$
$$0=(f)(x+x')+2(f^2)y-(fj)-(fj'),$$
$$0=n(x-x')+(j')-(j);$$

and the substitution of the numerical values gives

$$x=+80·03,$$
$$x'=+83·88,$$
$$y=-\ 0·593.$$

If we assumed as true the equation $K'_{1.2}+K'_{2.3}=K'$, we should arrive at the result,

$$K'=2K_{\frac{1}{2}}+163·91:$$

but this is not so; for an examination of the alignment of the dots shows that the centre dot is considerably misplaced. In order to measure the quantity, a very fine silk thread was stretched tightly just above the surface and close to the dots. Measurements were then made, and with this result, that the centre dot is 553 micrometer-divisions (one division$=\frac{1}{12400}$ inch) off the line joining the *end dots* of the klafter: it is also very slightly *above* the line joining the end dots; but this deviation is much smaller, and not large enough to produce any error. Now the 553 divisions correspond to about 1240 millionths of a yard. If we write δ for this deviation, it is clear that the distance of the extreme dots is less than the sum of the two portions by

$$\frac{\delta^2}{\frac{1}{2}K'}$$

where K is the length of the klafter—that is to say, a correction of

$$-\frac{1240^2}{1036942}=-1\cdot48$$

has to be applied to $x+x'=163\cdot91$. We have therefore finally this result, that the Pulkowa klafter exceeds two lengths of the half klafter by 162·44, or

$$K'=2K_{\frac{1}{2}}+162\cdot44\pm0\cdot38.$$

The Milan Copy of the Klafter.

The Milan copy of the Austrian klafter, which is deposited in the Observatory of Brera, in charge of Dr. SCHIAPARELLI, was, at the instance of Professor CHARLES LITTROW, Director of the Observatory at Vienna, borrowed by the Imperial Academy of Sciences of Vienna from the Italian Government, and sent to Southampton, under charge of Dr. EDMUND WEISS, to be compared with the yard and other standards. We shall denote this bar by the symbol K''. It is a bar of soft Styrian iron, 0·86 inch in breadth and 1·31 in depth. The upper surface has inlaid plates of platinum at its extremities, carrying dots, the distance between them being the klafter. At the left end of the bar is the inscription "K. K. Polytechn. Institut in Wien 1856," and in the centre "Wiener Klafter bei +13° Réaum. (Mass. der Punkte)." The lower surface has three dots, on inlaid platinum plates, marking the klafter and its subdivision into halves. The dots on the upper surface are marked I, II, those on the lower surface are marked 1, 2, 3. The bar is supported on points about one fourth and three fourths of its length; as in the Pulkowa klafter, small brass pieces or feet are screwed to the bar to insure an invariability in the supporting pressures.

The dots on this bar are very large, and this circumstance introduces a considerable amount of discord in the observations. The centre dot 2 is the largest.

Between April 17th and 22nd ten comparisons were made in the same manner as for the Pulkowa klafter, at a temperature of about 50°, and fourteen between July 6th and 12th, the temperature varying from 60°·7 to 62°·5.

The observations being reduced in the same manner as described for the Pulkowa klafter, give

$$K''_{1.2}=K_{\frac{1}{2}}+66\cdot73,$$

$$K''_{2.3}=K_{\frac{1}{2}}+72\cdot91.$$

Hence, at the temperature of 61°·25, the Milan klafter exceeds two lengths of the Ordnance half klafter by 139·64. The probable error of this determination is

$$\pm0\cdot865\sqrt{\cdot4205}=\pm0\cdot561.$$

This is a large quantity, and is due, in great measure, to the large size of the dots. The extreme dots 1, 3 on this bar measure about 115 divisions in diameter, and the central dot 2 measures 135 divisions, which is equivalent to 107 millionths of a yard.

Comparison between the Pulkowa and Milan Klafters.

The Milan klafter was compared with the Pulkowa copy in May, June, and July, 1869. Both surfaces of the former were compared—that is, both the distance [1 . 3] and the distance [I . II]. The klafter distance [1 . 3] was compared with the Pulkowa klafter ten times between May 3rd and 6th, at a temperature of about 53°—five times on June 12th and 14th, at about 58°—and five times on July 3rd and 5th, at near 60°.

The distance [I . II] was compared with the Pulkowa klafter eleven times between May 7th and 11th at a temperature of about 53°, and nine times on June 10th and 11th at about 58°.

The reduction of these observations by the method of least squares shows that the probable error of one comparison was

$$\pm 0{\cdot}674\sqrt{\frac{95{\cdot}989}{40-3}}=\pm 1{\cdot}08,$$

which is somewhat large; and for final results at 61°·25,

$$K'=K''_{1.3}+20{\cdot}86\pm 0{\cdot}43,$$
$$K'=K''_{\text{I.II}}+\ 5{\cdot}52\pm 0{\cdot}43.$$

We have now determined directly the lengths of the Pulkowa and Milan klafters, and also by direct observations their difference. It remains to examine as to the consistency of these results; they are, in fact, these:—

$$K'\ -2K_{\frac{1}{2}}\ =162{\cdot}44\pm 0{\cdot}38,$$
$$K''_{1.3}-2K_{\frac{1}{2}}\ =139{\cdot}64\pm 0{\cdot}56,$$
$$K'\ -\ K''_{1.3}=\ 20{\cdot}86\pm 0{\cdot}43,$$

which are not consistent, as the difference of the first and second exceeds the third by 1·94. If we adjust the three results by least squares, we get

$$K'\ =2K_{\frac{1}{2}}+162{\cdot}01,$$
$$K''_{1.3}=2K_{\frac{1}{2}}+140{\cdot}59.$$

But these results are not perfectly satisfactory, as from the manner in which they have been derived we cannot express their probable errors. In order to do this, it is necessary to reduce in one system of equations, by the method of least squares, all the following comparisons, namely:—

27 between $K'_{1.2}$ and $K_{\frac{1}{2}}$,

27 between $K'_{2.3}$ and $K_{\frac{1}{2}}$,

24 between $K''_{1.2}$ and $K_{\frac{1}{2}}$,

24 between $K''_{2.3}$ and $K_{\frac{1}{2}}$,

20 between $K'_{1.3}$ and $K''_{1.3}$,

20 between $K'_{1.3}$ and $K''_{\text{I.II}}$,

where it is to be remembered that $K'_{1.2}+K'_{2.3}=K'+1{\cdot}48$. Now let

$$\begin{aligned} K'_{1.2}-K_{\frac{1}{2}} &= u+fy, \\ K'_{2.3}-K_{\frac{1}{2}} &= v+fy, \\ K''_{1.2}-K_{\frac{1}{2}} &= u'+f'y', \\ K''_{2.3}-K_{\frac{1}{2}} &= v'+f'y', \\ K''_{\mathrm{I.II}}-K_{1.3} &= x. \end{aligned}$$

Then, further, if K' and $K''_{1.3}$ be compared at the temperature $61^{\circ}{\cdot}25+e$,

$$K'-K''_{1.3}=u+v-u'-v'+2ey-2ey'-1{\cdot}48\,;$$

and if K' and $K''_{\mathrm{I.II}}$ be compared at temperature $61^{\circ}{\cdot}25+e$,

$$K'-K''_{\mathrm{I.II}}=u+v-u'-v'-x+2e'y-2e'y'-1{\cdot}48.$$

Thus we have a system of 142 equations to solve, which finally give a group of seven equations resulting in the following quantities:—

$$\begin{aligned} u &= +79{\cdot}97, \\ v &= +83{\cdot}81, \\ u' &= +67{\cdot}08, \\ v' &= +73{\cdot}26, \\ x &= +15{\cdot}28, \\ y &= -\ 0{\cdot}6115, \\ y' &= -\ 0{\cdot}0759\,; \end{aligned}$$

and consequently at $61^{\circ}{\cdot}25$ the lengths of the klafters—

$$\begin{aligned} K' &= 2K_{\frac{1}{2}}+u+v-1{\cdot}48 = 162{\cdot}30, \\ K''_{1.3} &= 2K_{\frac{1}{2}}+u'+v' = 140{\cdot}34, \\ K''_{\mathrm{I.II}} &= 2K_{\frac{1}{2}}+u'+v'+x = 155{\cdot}62. \end{aligned}$$

The reciprocals of the weights of the determinations are

$$\begin{aligned} &{\cdot}0624, \\ &{\cdot}0892, \\ &{\cdot}1426. \end{aligned}$$

The sum of the squares of the 142 errors is 326·82; consequently the probable error of a single equation is $\pm 1{\cdot}05$, and the probable errors of the determinations of K', $K''_{1.3}$, and $K''_{\mathrm{I.II}}$ are respectively,

$$\begin{aligned} K' \quad \ldots\ldots \quad & \pm 1{\cdot}05\sqrt{{\cdot}0624}=\pm 0{\cdot}26, \\ K''_{1.3} \quad \ldots\ldots \quad & \pm 1{\cdot}05\sqrt{{\cdot}0892}=\pm 0{\cdot}31, \\ K''_{\mathrm{I.II}} \quad \ldots\ldots \quad & \pm 1{\cdot}05\sqrt{{\cdot}1426}=\pm 0{\cdot}40. \end{aligned}$$

If in the results we have now obtained we substitute the value of $2K_{\frac{1}{2}}$, we get finally for the Pulkowa copy of the klafter,

$$K' = 2{\cdot}07403658\mathfrak{P} \pm {\cdot}00000041;$$

and for the two lengths of the Milan copy,

$$K''_{1.3} = 2{\cdot}07401462\mathfrak{P} \pm {\cdot}00000045,$$
$$K''_{\text{I.II}} = 2{\cdot}07402990\mathfrak{P} \pm {\cdot}00000051.$$

In a pamphlet by M. STRUVE, entitled "Vergleichungen der Wiener Masse mit mehreren auf der Kaiserl. russischen Hauptsternwarte zu Pulkowa befindlichen Masseinheiten," 1850, we find it stated that the length of the Pulkowa copy of the Austrian klafter, as determined by Professor STAMPFER, is

$$0^{\text{l}}{\cdot}00029 \pm 0^{\text{l}}{\cdot}00020$$

shorter than the legal or standard Vienna klafter, the unit here being the "line" or $\frac{1}{864}$ part of the klafter, that is 2400 millionths of a yard. Hence if $\mathfrak{K}$ be the true length of the klafter, the Pulkowa copy at 61°·25 is

$$K' = \mathfrak{K} - 0{\cdot}70 \pm 0{\cdot}48,$$

the unit here being, as usual, the millionth of a yard. The date of the certificate is April 1849.

Professor STAMPFER also compared the Milan copy of the klafter with the legal or standard Vienna klafter, and with this result, dated October 1856,

$$K''_{1.3} = \mathfrak{K} - 0^{\text{l}}{\cdot}0058 \pm 0{\cdot}0005 \quad . \; . \; . \quad \text{10 comparisons,}$$
$$K''_{\text{I.II}} = \mathfrak{K} + 0^{\text{l}}{\cdot}0000 \pm 0{\cdot}0004 \quad . \; . \; . \quad \text{10 comparisons,}$$

the bar being at 61°·25. This, expressed in millionths of a yard, is

$$K''_{1.3} = \mathfrak{K} - 13{\cdot}92 \pm 1{\cdot}20,$$
$$K''_{\text{I.II}} = \mathfrak{K} - 0{\cdot}00 \pm 0{\cdot}96.$$

It appears from this that the difference of the klafters on the two surfaces of the Milan copy, as determined at Southampton and at Vienna, are in tolerable accordance, being 15·28 in the one case and 13·92 in the other.

But the discordance between the Milan and Pulkowa copies is considerable. Taking the length I.II for example, the difference of the two bars as determined at Southampton is 6·68, while the Vienna comparisons imply a difference of −0·70.

But it is difficult to imagine any constant error in the observations made here, as not only the frequent alterations of the adjustments and daily shiftings of the bars makes this very improbable, but an examination of the tables of errors of the different series of comparisons shows no trace of such error.

Vienna Toise.

The toise marked on the Pulkowa klafter was compared twelve times with the Ordnance Survey toise in September 1867 at temperatures from 58° to 59°, and nine times in December of the same year, the temperature ranging from 39° to 42°.

The observations being reduced by the method of least squares lead to the following result, both bars being at the temperature of 61°·25,

$$T_V = T_0 - 362{\cdot}73 \pm 0{\cdot}27,$$

where T_V is the length of the Vienna toise, and T_0 that of the Ordnance toise.

Now the length of the T_0 at 61°·25 is, as given at page 280 of the 'Comparisons of Standards,'

$$T_0 = 2{\cdot}13166458\,\mathfrak{P}.$$

Hence the length of the toise on the Vienna klafter

$$T_V = 2{\cdot}13130185\,\mathfrak{P}.$$

If we compare this with the length of the Prussian toise T_{10}, which (Comp. Stand. page 280) was found by comparisons to be

$$T_{10} = 2{\cdot}13150911\,\mathfrak{P},$$

it appears that the Vienna copy is shorter than the Prussian by 207·26 ; that is,

$$T_V = T_{10} - 207{\cdot}26.$$

The lengths obtained from these two toises by M. STRUVE, see 'Arc du Méridien . . . 1860,' vol. i. p. lxxii, are (he calls the bar B′)

$$T_{10} = 863^{l}{\cdot}99914,$$

$$T_V = 863{\cdot}91726\,;$$

and the difference of these expressed in millionths of a yard is

$$T_V = T_{10} - 202{\cdot}00.$$

The agreement of this with the Southampton results is not very satisfactory.

The ratio of the klafter to the toise as both marked on the Pulkowa klafter is, according to the observations we have recorded,

$$\frac{2{\cdot}07403629}{2{\cdot}13130185} = {\cdot}973130850.$$

The same ratio, according to the measures of M. STRUVE, is

$$\frac{840{\cdot}70342}{863{\cdot}91726} = {\cdot}973129556.$$

The same ratio as obtained from the observations of M. STAMPFER is (see 'Vergleichungen der Wiener Masse &c.,' page 15)

$$0{\cdot}97313204.$$

The ratio as determined in the observations at Southampton is therefore intermediate between these.

Standard Bar of the Cape of Good Hope.

The 10-feet Cape standard is a bar similar in construction to the Ordnance 10-feet standard. In April 1844 one hundred comparisons were made between these bars at Southampton, giving the difference of length

$$O_1-B=50\cdot08.$$

These comparisons were renewed in January 1868 and September 1869, fifteen being made on six days in the former month at a temperature of about 41°, and fifteen on five days in the latter month at about 59°·5.

The reduction of the observations by the method of least squares gives the difference of length at 62°,

$$O_1-B=51\cdot12\pm0\cdot25.$$

The agreement of this with the comparisons of 1844 gives much confidence in the stability of these bars, and is almost as remarkable as the result obtained from the Indian bar I_b (see 'Comparisons of Standards,' page 255).

The Spanish Standard.

The iron four-metre bar belonging to the Spanish Government, which was sent to this country in the autumn of 1869, is a copy of the geodetic standard of Spain. This standard itself was constructed from BORDA's double-toise No. 1, of which the length is 3·8980732* metres. The description of this standard, and a very full and elaborate exposition of the methods of comparison, is given in the work entitled "Expériences faites avec l'appareil à mesurer les bases appartenant à la Commission de la Carte d'Espagne" (Paris, 1860).

This bar is constructed of two plates of iron in the form of a ⊥, the upright plate being nearly 5 inches in depth, and the horizontal plate 3½ inches in width. These plates are connected by thirteen pairs of angle-pieces. Four thermometers, which are Centigrade and of very excellent workmanship, are attached at the sides of the bar, being of course removable at pleasure. A levelling-apparatus is attached at the upper edge of the bar. The bar is lifted by two pairs of handles; it is not enclosed in a box when in use as a measure, and has no other protection than paint. For travelling it is protected in a very strong box.

On the upper edge of the bar are inlaid five small disks of platinum, and on these are drawn fine lines which mark off the lengths of four metres. The part of each line to be observed is the extremity contiguous to the edge of the bar. The lines are unfortunately not very fine, though otherwise very well drawn.

The following is the certificate of this bar:—

"*Règle en fer appartenant au Gouvernement Espagnol.* Pour chaque degré du thermomètre centigrade elle se dilate de

$$0^{mm}\cdot043193\pm0^{mm}\cdot000009,$$

elle a été comparée avec le règle en platine, étalon géodésique de l'Espagne, lequel étalon

* Base du Système Métrique Décimal, tome iii. pp. 139, 228.

l'a été aussi directement avec la règle No. 1 de BORDA déposée, à l'observatoire de Paris. De ces comparaisons il résulte que la règle en fer a de longueur à la température de 21°·935 centigrades,

$$4^{m}\cdot 0006526 \pm 0^{m}\cdot 000002.$$

"C. IBAÑEZ,
Colonel du génie, attaché aux travaux géodésiques de l'Espagne."

This bar has been compared in two entirely different manners—first, with two lengths of the Ordnance toise, and secondly with four lengths of the Ordnance metre. The former comparison was rendered possible by the fortunate circumstance that four metres exceed two toises by almost exactly 4 inches, the difference (·013 inch) being easily measured by the micrometers.

Comparisons of the Spanish Bar with the Toise.

In order to effect this comparison the four micrometer-microscopes H, I, J, K were arranged in line at the distances H I=one toise, I J=4 inches, J K=one toise, so that H K=four metres very approximately. The adjustment into a horizontal straight line of the outer foci of these microscopes was a matter of some difficulty, but was finally effected with so much precision that not the smallest error in the comparisons could arise therefrom. It remains to add that the alignment was frequently examined during the course of the observations.

The method of comparison will be understood from the following Table, where the different operations are on successive lines: τ_1 is the mean of the readings of the four thermometers of the Spanish bar before the micrometers are read, e and f are the readings of the Spanish bar by the micrometers H K, τ_2 the temperature of the bar immediately after, and so on. The toise and foot were mounted side by side in the same box. Every reading was made by two observers independently.

Bars.	Microscopes.				Thermometers.
	H	I	J	K	
					τ_1
Spanish . . .	e	...	...	f	
					τ_2
					t_1
Toise	a	b			
Foot	...	b'	c		
Toise	...	...	c'	d	
					t_2

Let the distances of the zeros of the four micrometers H, I, J, K be

$$\text{HI}=\text{P} : \text{IJ}=\text{Q} : \text{JK}=\text{R}.$$

Then, remarking that the screw-heads of H and I were to the left, and those of J K to the right, we have

$$\begin{aligned} \text{P}+\text{Q}+\text{R}&=\text{S}+eh+fk, \\ \text{P}&=\text{T}+ah-bi, \\ \text{Q}&=\text{F}+b'i+cj, \\ \text{R}&=\text{T}-c'j+dk, \end{aligned}$$

where h, i, j, k are the values of one division of the respective microscopes. Hence we get

$$\text{S}=2\text{T}+\text{F}+(a-e)h+(b'-b)i+(c-c')j+(d-f)k,$$

where S, T, F represent the length of the Spanish bar, the toise, and the 4-inch space on the foot respectively. This expression gives us the actual difference of length of the bars at the time of comparison. The foot being in the same box with the toise, it is assumed that its temperature is the same as that of the toise.

The comparisons of the Spanish bar with the toise extend over seven days in October, three days in November, and three days in December 1869, in all twenty-three comparisons. The readings of one observer lead to the result, the bars being all at 61°·25,

$$\text{S}=2\text{T}_0+4\text{I}+499{\cdot}63,$$

and those of the other to the result,

$$\text{S}=2\text{T}_0+4\text{I}+499{\cdot}58;$$

and taking the means of the corresponding readings of the two observers, the result is

$$\text{S}=2\text{T}_0+4\text{I}+499{\cdot}60\pm0{\cdot}23.$$

The 4-inch space 4I actually used was the space $[c\,g]$. The value of this space is, see page 450, at the temperature of 61°·25,

$$[c\,g]=\tfrac{1}{9}\text{Y}_{55}-2{\cdot}73\pm0{\cdot}06.$$

Consequently

$$2\text{T}_0+[c\,g]=4{\cdot}37443869\text{Y}_{55}\pm0{\cdot}42;$$

so that finally we have, the Spanish bar being at 61°·25 and the yard at 62°,

$$\text{S}=4{\cdot}37493829\text{Y}_{55}\pm0{\cdot}48.$$

Comparisons with four lengths of the Ordnance Metre.

In order to effect this, five microscopes were adjusted in line at one metre apart. Ten comparisons were made in October and November 1869. The readings of one observer lead to the result,

$$\text{S}=4\text{M}_0-58{\cdot}81,$$

those of the second observer to the result,

$$S=4M_0-60\cdot62;$$

and, taking the means of the corresponding readings of the two observers,

$$S=4M_0-59\cdot72\pm0\cdot65,$$

both bars being at 61°·25.

Now the ratio of $4M_0$ at 61°·25 to Y_{55} at 62° is

$$4M_0=4\cdot377499376Y_{55}\pm0\cdot64;$$

consequently the Spanish bar at 61°·25 is

$$S=4\cdot37493404Y_{55}\pm0\cdot91.$$

The two results we have now arrived at as to the length of S differ by 4·25±1·03. If we combine them with regard to their probable errors, we get finally

$$S=4\cdot37493737Y_{55}.$$

By page 280 of the 'Comparisons of Standards,' it appears that at 62°

$$Y_{55}=\cdot91439143\,\mathfrak{M};$$

whence the length of the Spanish bar is, expressed in metres,

$$S=4\cdot00040524\,\mathfrak{M}.$$

We now compare this with the result obtained by Colonel IBAÑEZ at Madrid; according to the certificate accompanying the bar, its length at the temperature of 21°·935 Centigrade is 4·0006526 metres, and the rate of expansion is ·000043193 metre for each degree Centigrade; hence the length of the bar at 61°·25 Fahrenheit, according to the Spanish observations, is

$$4\cdot0004071\,\mathfrak{M}.$$

From this it appears that the length of the metre, as given by the Spanish bar, agrees with great precision with that inferred at Southampton from the comparisons of the Belgian and Prussian toises.

The difference between the two lengths obtained for the Spanish bar, first through the toise, and secondly through the metre, suggested the direct comparison of those bars one with the other by means of the approximate relation

Toise+2 inches=2 metres.

For this purpose five microscopes were adjusted in line in the order H, K, G, I, J, at the distances H G=G I=one metre, H K=5 inches, I J=3 inches. Then the comparison was effected by observing in succession:—

1st, the Ordnance toise under K and J;
2nd, the Ordnance metre under G and I;
3rd, the Ordnance metre under H and G;
4th, 5-inch space [*af*] on foot under H and K;
5th, 3-inch space [*a d*] on foot under I and J.

Hence we have immediately the value of $2M-T-[df]$ at the temperature observed. Ten comparisons were made on the 4th, 5th, and 6th of November, 1869, at temperatures between 50°·73 and 49°·83. The mean of the observations gave

$$2M-T-[df]=277{\cdot}10\pm0{\cdot}25$$

at the temperature of 50°·33. Now, referring to pages 144, 69, and 110 of the 'Comparisons of Standards,' we find that at the temperature of 50°·33

$$2M-T-[df]=276{\cdot}12\pm0{\cdot}38.$$

The agreement of results so totally independent of one another, and each obtained through a very complicated system of observations, must be considered very satisfactory.

Comparisons of the American Metre with the Ordnance Survey Metre.

In the Appendix No. 26 of the United States Coast Survey Report for 1862 will be found, in connexion with the experiments for determining the lengths of the six-metre standard bar and its rate of expansion, mention of six iron metres designated No. 1, No. 2, . . . No. 6, which were compared with the "Committee Metre," or supposed standard metre. These bars are all end measures; the section is a rectangle of 1·13 inch by 0·27 inch.

The unit of length to which all distances measured in the Coast Survey are referred is the French *metre*, an authentic copy of which is preserved in the archives of the Coast Survey Office. It is the property of the American Philosophical Society, to whom it was presented by Mr. HASSLER, who had received it from TRALLES, a member of the French Committee charged with the construction of the standard weights and measures according to the decimal system. This metre is of iron, and was one of the twelve original bars used in the construction of the standard metre by comparison with the *toise*, which had served as unit of length in the measurement of the meridional arcs in France and Peru. It possesses all the authenticity of any original metre extant, bearing not only the stamp of the Committee, but also the original mark (·:) by which it was distinguished from the other bars during the operation of standarding. It is always designated as the *Committee metre*.

The iron metres have in all comparisons been placed on edge, and supported at one fourth and three fourths of their length. The metre compared at Southampton was that marked No. 6.

The length of this metre increased by the two parts of the contact-apparatus was compared with the Ordnance metre. Twenty-eight comparisons were made in January and February 1868, eight in October of the same year, and ten in May 1869. Reducing the observations by least squares, we find the probable error of a single comparison

$$\pm0{\cdot}771;$$

and finally, both bars being at 32°, M_a representing the American metre,

$$M_a=M_0+44{\cdot}37\pm0{\cdot}33,$$

or, both being at 62°,

$$M_a = M_0 + 54{\cdot}55 \pm 0{\cdot}37.$$

It is unfortunate for the success of these comparisons that the expansion of neither of the metres is known by direct experiment. We can arrive at a value of the expansion of M_0 through its comparison with Y_{55}, but the expansion of Y_{55} is known only indirectly. This last bar has been compared with two iron bars, each of which has had its expansion determined by direct experiment. The first of these bars is the Indian 10-feet steel standard. The expansion of Y_{55}, as derived from this source, is

$$6{\cdot}650.$$

But the expansion of this same yard, as inferred from the 10-feet bar OI_1 (see 'Comparisons of Standards,' pages 90 and 227), is

$$6{\cdot}514.$$

The discrepancy between these two is very considerable, especially when, as in the present case, it has to be multiplied by 30. We have, however, no alternative but to adopt the mean as the expansion of Y_{55}, namely 6·582. By page 106 of the 'Comparisons of Standards,' the expansion of a yard of the Ordnance metre is less than this by 0·411; that is, the expansion of the metre for 30° Fahr. is

$$6{\cdot}171 \times 30 \times 1{\cdot}09375 = 202{\cdot}48.$$

Consequently the length of the metre at 32° (see 'Comparisons of Standards,' p. 110)

$$= 1{\cdot}09355096 Y_{55},$$

with a probable error of perhaps ·00000250.

Thus the length of the American metre at 32° is (Y_{55} being at 62°)

$$1{\cdot}09359533 Y_{55}.$$

We may arrive at the length of the American metre otherwise than by using the inferred expansion of the Ordnance metre. For it is stated, in the Report referred to, that in the comparisons of the sum of the six iron metres with the six-metre standard at different temperatures, no variation in the difference of length was found corresponding to different temperatures; that is, the expansion of the iron metres was equal to the expansion of the standard six-metre bar. Now the coefficient of the expansion of the standard was found by very careful experiments to be

$$\cdot00000641.$$

Hence the actual expansion of one of the metres between 32° and 62° would be

$$30 \times 6{\cdot}41 \times 1{\cdot}0936 = 210{\cdot}30.$$

Hence M_a being at 32°, and M_0 at 62°,

$$M_a = M_0 + 54{\cdot}55 - 210{\cdot}30$$
$$= 1{\cdot}09359769 Y_{55}.$$

This differs but 2·36 from our former result. The mean of the two is

$$M_a = 1{\cdot}09359651 Y_{55},$$

the probable error of which we may well assume to be about $\pm 2{\cdot}00$.

The length of the American metre No. 6, according to the Report, Appendix No. 26, is, at 32°, less than the "Committee metre" by 3·41 fifteen thousandths of an inch, that is

6·31.

By more recent comparisons made in March 1869 (kindly communicated by J. E. HILGARD, Esq.), No. 6 is short by 3·2 ten thousandths of an inch, that is

8·89.

Taking the mean of these, we infer the length of the American standard metre to be

$$1{\cdot}09360411 Y_{55}.$$

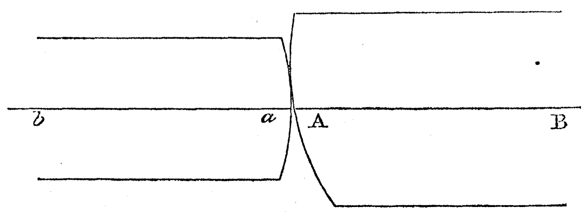
The Report referred to mentions that "besides the Committee metre there is in the collection of the Coast Survey another iron metre of special value, having been made by LENOIR, the artist who performed the mechanical operations in the preparation of the original metres. It is designated as the "LENOIR metre," and has been found, by accurate comparisons with the Committee metre, to be shorter than the latter by $0^{m}{\cdot}0000258$." This quantity is 28·21 millionths of a yard. Hence the LENOIR metre has for length,

$$\text{LENOIR metre} = 1{\cdot}09357590 Y_{55}.$$

The Russian Double Toises.

The Russian Double Toise P, of which observations are now to be recorded, is the same bar that was compared at Southampton in 1865. In those comparisons the woollen packing which surrounded the bar was allowed to remain undisturbed. It appears, however, from a subsequent communication from M. STRUVE, that the bar has been accustomed to be compared with the packing removed, and only a thin covering of cloth remaining wrapped round it. It was therefore judged expedient by M. STRUVE that the bar should be returned to Southampton for further comparisons to be made without the woollen packing. Moreover, in the comparisons of this bar made in Russia, the contacts are said to have been made at the centres of the disks, and not at the points corresponding to the maximum of length, at which the contacts were made here.

The small terminal cylinder of this bar has a diameter of 0·25 inch, and the centre of curvature of the terminal surface, instead of being, as it should be, in the axis of the cylinder, is 0·025 inch above it. Let A B be the horizontal axis of the cylinder, $a\,b$ that



of the cylinder of another bar brought up to contact. Suppose that the centre of the curved surface of contact of the second cylinder is *in* its axis $a\,b$, and suppose r to be the radius of the spherical surface, R being the radius of the corresponding spherical terminal surface of the cylinder A B of the bar P.

Suppose, in the *first* case, the bars to be so adjusted that the axis $a\,b$ is in the same line with the axis A B. In this case the length of the horizontal projection of the line joining the centres of the spherical surfaces is

$$=\mathrm{R}+r-\frac{\frac{1}{2}e^2}{\mathrm{R}+r},$$

where e is the distance of the centre of the terminal spherical surface of the cylinder above its axis A B, or the distance from A B of that point of the terminal surface at which the normal to that surface is horizontal. But if the cylinder $a\,b$, still maintaining contact, were raised by the quantity e, the common tangent plane at the point of contact would be vertical, and the distance of the centres of the spherical surfaces would be $r+\mathrm{R}$. Hence the bar is apparently shortened by the quantity

$$\frac{1}{2}\frac{e^2}{r+\mathrm{R}}.$$

Secondly, for the case of an actual contact in the centre of the terminal disk of the cylinder of P or in the axis A B, let the axis $a\,b$ be below A B by the quantity

$$\frac{re}{\mathrm{R}}.$$

In this case the contact is in the actual centre of the terminal disk of the bar P, and the horizontal projection of the line joining the centres is

$$\mathrm{R}+r-\frac{1}{2}\cdot\frac{\mathrm{R}+r}{\mathrm{R}^2}e^2;$$

so that the error in this case is

$$\frac{1}{2}\cdot\frac{\mathrm{R}+r}{\mathrm{R}^2}e^2.$$

To reduce these to numerical results for the bar P, the value of R is 2·00 inches, $e=\frac{1}{40}$ inch. For the contact-apparatus the radius $r=0{\cdot}75\pm{\cdot}03$ inch, obtained, as in the case of the bar P, by the measurement of optical images formed by reflection. Expressed in millionths of a yard, one inch=27778; hence, since the same defect exists at each end of P:—

Case 1. When the axes of the cylinders are at the same height,

$$\text{Correction}=\frac{e^2}{r+\mathrm{R}}=6{\cdot}31;$$

Case 2. When the contact is at the centre of the disk of P,

$$\text{Correction}=\frac{(r+\mathrm{R})e^2}{\mathrm{R}^2}=11{\cdot}93.$$

The correction in the latter case is nearly double that in the former, while the actual difference in position of the points of contact in the two cases is but $\frac{1}{133}$ of an inch.

If P were placed in contact with other bars having terminal cylinders of the same diameter and same radius of curvature for the spherical surfaces, but free from the peculiar defect of P, the correction would be, supposing the axes adjusted into the same line, or the touching cylinders at the same height,

$$\frac{e^2}{2R}=4\cdot34.$$

If the contacts at the two ends of P were made with very small cylinders, the radius of curvature of whose spherical contact surfaces were *exceedingly small*, then, supposing the axes of these cylinders to be coincident with the axes of the corresponding cylinders of P, the correction to the length of P would be

$$\frac{e^2}{R}=8\cdot68;$$

and this quantity is equal to the difference between the maximum length of P and the distance between the centres of its terminal disks.

The comparisons of P in 1865 were commenced, as stated at page 258 of the 'Comparisons of Standards,' in ignorance of the peculiarity of these terminal surfaces. It was attempted to make contacts at the centres of the disks, while at the same time the axes of the cylinder of P and of the contact-apparatus were in one line. It may be assumed that the result would be less than the maximum length of the bar by some quantity intermediate to the two corrections (Case 1 and Case 2) which we have just computed, viz. 6·31 and 11·93. A reduction of the forty observations so made shows the length obtained to have been 10·15 shorter than that which resulted afterwards, when the contacts were made so as to give the maximum length. So far this is perfectly satisfactory.

The new comparisons of P extend over five days in March 1868 and one day in April, the temperature being about 45°; over nine days in August and one in September 1869, temperatures ranging from 61° to 65°. In these comparisons the contacts were so adjusted that the axes of the cylinders of P and of the contact-apparatus were in the same line (Case 1).

The result of the forty comparisons is this, both bars being at 61°·25,

$$P=2T_0-323\cdot90\pm0\cdot29.$$

If we correct the observations of 1868–69 for the position of the point of contact, so as to give the maximum length of the bar, we get, by adding 6·31,

$$P=2T_0-317\cdot59.$$

This differs sensibly from the length determined in 1865, which was

$$P=2T_0-320\cdot48.$$

If from the maximum length we would determine the length of the bar measured between the centres of the terminal disks, we have to subtract 8·68; this gives

$$P=2T_0-326\cdot27.$$

The Double Toise Q.—This is a bar similar in every respect to P, but without the defect in the terminal surfaces. Eighteen comparisons were made in March 1868 at temperatures from 45° to 46°, and twenty-two comparisons in September 1869 at temperatures between 58° and 61°. These observations, reduced by least squares, give the result,

$$Q=2T_0-360{\cdot}00\pm0{\cdot}40;$$

and this, when compared with the length we have determined for P, viz. $P=2T_0-323{\cdot}90$, is in tolerable accordance with M. STRUVE's determination of the lengths of these bars. According to M. STRUVE, $P-Q=0^{l}{\cdot}01370$, which, expressed in millionths of a yard, is

$$P-Q=33{\cdot}80,$$

while the difference, as determined at Southampton, is

$$P-Q=36{\cdot}10.$$

But there is not the same accordance if we compare the difference between Q and two lengths of the Prussian toise denoted B′. The length of 2B′ (STRUVE, 'Arc du Méridien . . . ,' tom. i. p. lxxiii) is

$$2B'=1727{\cdot}99828;$$

and this exceeds the length of Q, namely 1727·97386, by $0^{l}{\cdot}02442$, which, expressed in millionths of a yard, is

$$2B'-Q=60{\cdot}25=2T_{10}-Q.$$

But, according to the observations made at Southampton ('Comparisons of Standards,' page 273),

$$2T_{10}=2T_0-309{\cdot}04,$$

which, compared with the length of Q determined above, gives

$$2T_{10}-Q=50{\cdot}96,$$

showing a discrepancy of 9·29 millionths of a yard, as though Q had increased in length between the comparisons at Pulkowa and at Southampton.

In order to throw some light upon this discrepancy, in the autumn of 1871, just two years after the comparisons of Q detailed in the preceding paragraph, some further observations were made of Q, the comparisons between it and two lengths of the Ordnance toise being made with the utmost care and circumspection, and the temperature being remarkably uniform during the whole time. The result of these ten comparisons was

$$Q=2T_0-355{\cdot}74\pm0{\cdot}33.$$

In combining our two results for the length of Q we shall not make use of their probable errors, as manifestly they are delusive on account of the existence of some constant source of error. If we are content to take the mean, it is

$$Q=2T_0-357{\cdot}87;$$

but to this we cannot assign its probable error.

If we compare this with our results for P, viz.

$$P=2T_0-323·90,$$

we get for the difference

$$P-Q=33·97,$$

which agrees at least very satisfactorily with the difference of length as determined at Pulkowa, viz. 33·80.

The length of the bar P, as adopted in the 'Comparisons of Standards,' was that assigned to it in STRUVE'S 'Arc du Méridien,' vol. i., viz. 1727·99440. But it appears that more recent comparisons at Pulkowa brought to light a decided alteration of length in P, and show that the length of P is to be taken as

$$P=1727·98756.$$

For the bar Q the length to be used is

$$Q=1727·97386.$$

Measures.	Standard Temperature.	Expressed in Terms of the Standard Yard. — Y.	Expressed in Inches. — Inch $=\frac{1}{36}$ Y.	Expressed in lines of the Toise. — Line $=\frac{1}{864}$ T.	Expressed in Millimetres. — Millimetre $=\frac{1}{1000}$ M.
	Fahr. Ther.				
The Yard		1·00000000	36·000000	405·34622	914·39179
Copy No. 55	62°·00	0·99999960	35·999986	405·34606	914·39144
Ordnance Half Klafter	61·25	1·03693714	37·329737	420·31855	948·16681
Klafter, Pulkowa Copy	61·25	2·07403658	74·665317	840·70290	1896·48203
Klafter, Milan Copy [I.3]	61·25	2·07401462	74·664526	840·69399	1896·46195
Klafter, Milan Copy [I.II]	61·25	2·07402990	74·665076	840·70019	1896·47592
Toise laid off on Pulkowa Copy of Klafter	61·25	2·13130185	76·726867	863·91516	1948·84492
Spanish Four-Metre Bar	61·25	4·37493562	157·497682	1773·36363	4000·40522
American Metre No. 6	32·00	1·09359607	39·369459	443·28504	999·97527
Russian Double Toise P, between the centres of terminal disks	61·25	4·26300289	153·468104	1727·99212	3898·05485
Russian Double Toise Q	61·25	4·26297129	153·466966	1727·97931	3898·02596
Cape 10-foot Standard	62·00	3·33328605	119·998298	1351·13491	3047·92941
The Klafter	......	2·07403483	74·665254	840·70219	1896·48043

	Inches.	Millimetres.
Royal Cubit	20·727	526·46
Common Cubit	18·240	463·29